Math 102

Krishanu Sankar

November 22, 2018

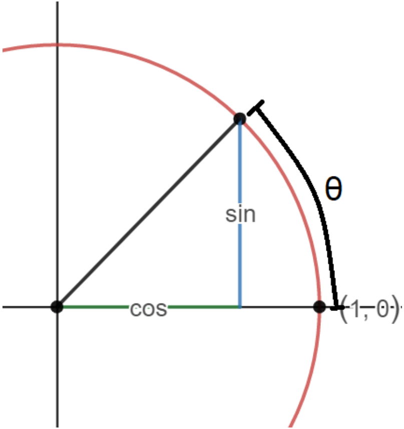
Announcements

- Office Hours Today: Pushed forward to 1:30 - 3:00PM.
- Review Sessions next week: sometime in the 4-7PM window next week Wednesday, Thursday, Friday. Exact times and locations TBD.

Goals Today

- Inverse Trig functions (Guest Lecture by Amin)
  - Review of inverse functions
  - Arcsin, arccos, arctan
  - Domain and range
- Fitting a sine function - practice

Recall...

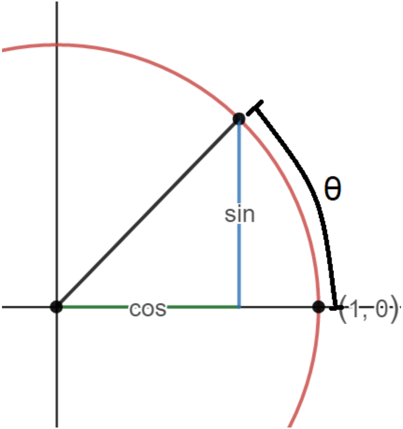


Consider a point that is a distance of θ along the circumference of the circle of radius 1 centered at $(0,0)$. (starting at $(1,0)$, going counterclockwise)

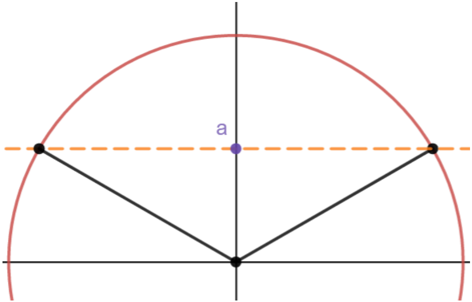
$$\sin(\theta) = y - \text{coordinate}$$

$$\cos(\theta) = x - \text{coordinate}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$



- $\sin(\theta)$: 'If I go a distance of θ around the circle, what is my y-coordinate?'
- $\sin^{-1}(a)$: 'If I am at y-coordinate a, then where on the circle am I?'



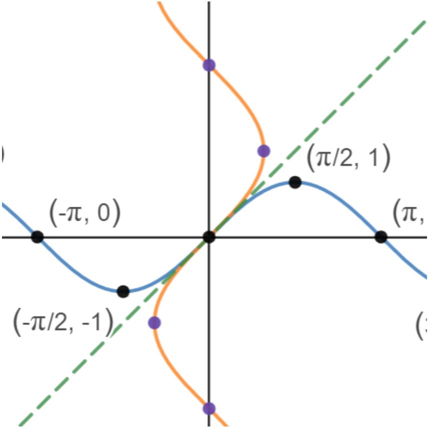
$\sin^{-1}(a)$: 'If I am at y-coordinate a, then where on the circle am I?'

Issue: $\sin^{-1}(a)$ is not unique!

Inverse functions

$y = \sin(x)$

$\sin(y) = x$

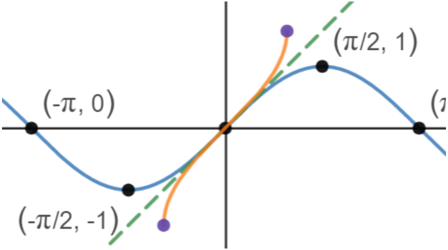


Same Issue: The graph of $\sin(y) = x$ does not pass the vertical line test.

Inverse functions

$y = \sin(x)$

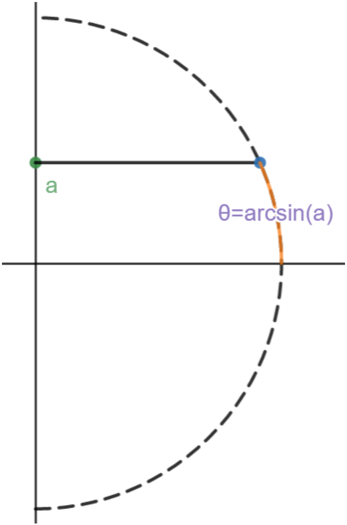
$y = \arcsin(x)$



Solution: Let's cheat. Cut the graph off at $(1, \frac{\pi}{2})$ and $(-1, -\frac{\pi}{2})$.

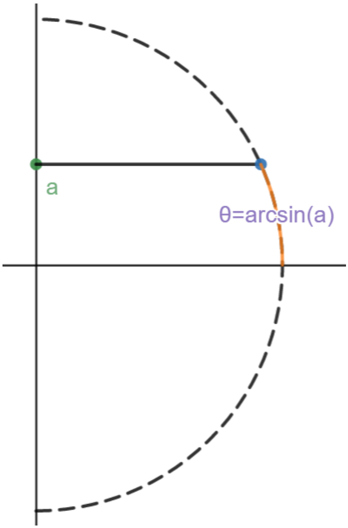
Definition: $\arcsin(x)$ is defined to be the unique number in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine equals x.

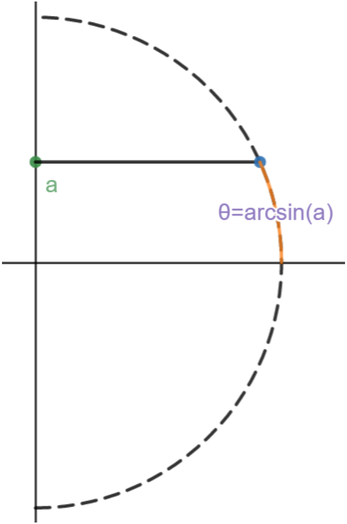
$\arcsin(x)$



- $\sin(\theta)$: 'If I go a distance of θ around the circle, what is my y-coordinate?'
- $\arcsin(a)$: 'If I am at y-coordinate a, and am somewhere on the circle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then where on the circle am I?'

Question: What is $\arcsin(\frac{1}{2})$?

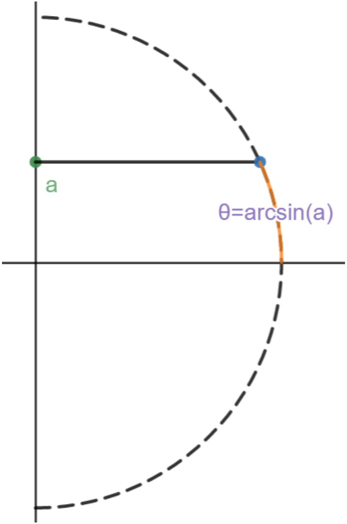




Question: What is $\arcsin(\frac{1}{2})$?

$\boxed{\arcsin(\dfrac{1}{2}) = \dfrac{\pi}{6}}$, because $\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.

Question: $\arcsin(\sin(\frac{5\pi}{2}))$?



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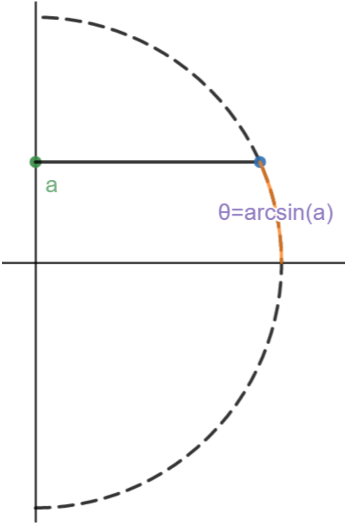
$\arcsin(\sin(\frac{5\pi}{2})) = \frac{\pi}{2}$, because $\frac{\pi}{2}$ and $\frac{5\pi}{2}$ have the same sine and $-\frac{\pi}{2} \leq \frac{\pi}{2} \leq \frac{\pi}{2}$.

Question: $\arcsin(\sin(\frac{2\pi}{3}))$?



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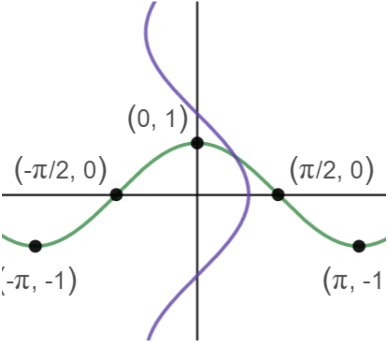
Inverse functions

$y = \cos(x)$

$\arccos(y) = x$



Same Issue: The graph of $\cos(y) = x$ does not pass the vertical line test.

Inverse functions

$y = \cos(x)$

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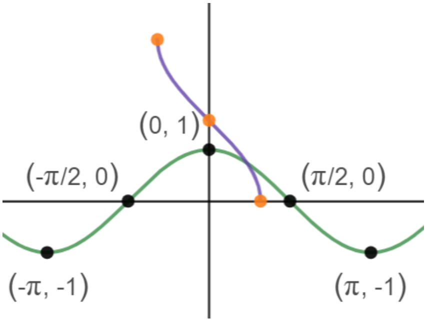
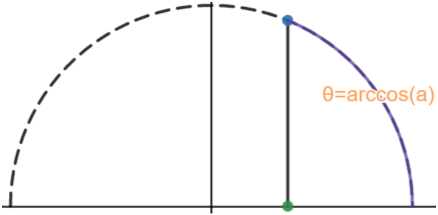


Solution: Cheat again. Cut the graph off at $(1, 0)$ and $(-1, \pi)$.

Definition: $\arccos(x)$ is defined to be the unique number in the range $[0, \pi]$ whose cosine equals x.



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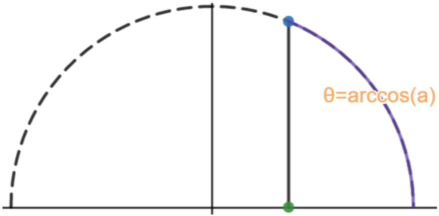

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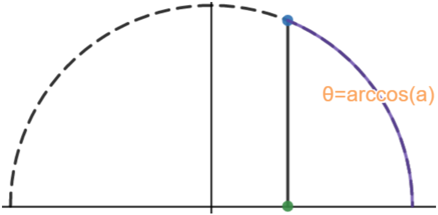
$\boxed{\arccos(\frac{1}{2}) = \frac{\pi}{3}}$
because $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $0 \le \frac{\pi}{3} \le \pi$.

Question: $\arccos(\cos(\frac{4\pi}{3}))$?



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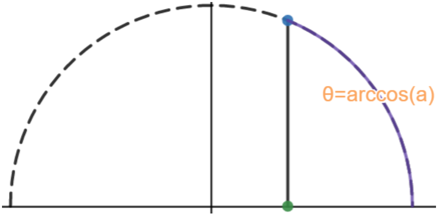
Question: $\arccos(\cos(\frac{4\pi}{3}))$?

$\boxed{\arccos(\cos(\frac{4\pi}{3})) = -\frac{2\pi}{3}}$, because $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ have the same cosine and $0 \leq \frac{2\pi}{3} \leq \pi$.

Question: $\arccos(\sin(-\frac{\pi}{4}))$?



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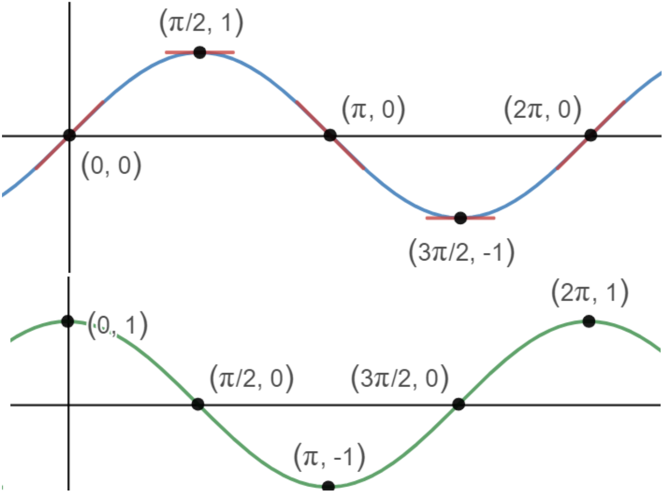
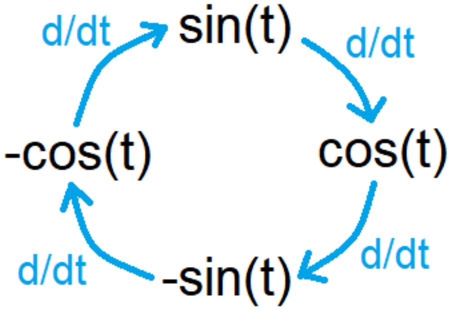
Question: $\arccos(\sin(-\frac{\pi}{4}))$?

$\arccos(\sin(-\frac{\pi}{4})) = \arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$.



- $\cos(t) = \sin(t + \frac{\pi}{2})$
- $\cos(t) = (\sin(t))'$ $\qquad -\sin(t) = (\cos(t))'$

The Circle of Sine



$$\cos(t) = \sin(t + \frac{\pi}{2})$$

$$-\sin(t) = \sin(t + \pi)$$

$$-\cos(t) = \sin(t + \frac{3\pi}{2})$$

Other Trig Derivatives

$$(\tan(t))' = \left(\frac{\sin(t)}{\cos(t)}\right)'$$

$$= \frac{(\sin(t))'\cos(t) - \sin(t)(\cos(t))'}{\cos^2(t)}$$

$$= \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} = \sec^2(t)$$

$$(\sec(t))' = \left(\frac{1}{\cos(t)}\right)' = -\frac{(\cos(t))'}{\cos^2(t)} = \frac{\sin(t)}{\cos^2(t)}$$

$$= \sec(t)\tan(t)$$

Adding sines and cosines

Consider the function

$$f(t) = \sin(t) + \cos(t)$$

Question: For what values of t is $f(t) = 0$? At what values of t does $f(t)$ achieve its minimum? Maximum?

Question: Can you write $f(t)$ in the form

$$f(t) = A\sin(t - \phi)$$

for some constants A, ϕ?

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$$f(t) = 0 \implies \sin(t) = -\cos(t) \implies t = 3\pi/4, 7\pi/4$$

Adding sines and cosines

Consider the function

$$f(t) = \sin(t) + \cos(t)$$

Question: For what values of t is $f(t) = 0$? At what values of t does $f(t)$ achieve its minimum? Maximum?

$$f'(t) = \cos(t) - \sin(t) = 0$$

$$\implies \sin(t) = \cos(t) \implies t = \pi/4, 5\pi/4$$

Check: $f(\pi/4) = \sqrt{2}$, $f(5\pi/4) = -\sqrt{2}$

Adding sines and cosines

Consider the function

$$f(t) = \sin(t) + \cos(t)$$

- Zeroes at $t = 3\pi/4$ and $t = 7\pi/4$.
- Maximum of $\sqrt{2}$ at $t = \pi/4$.
- Minimum of $-\sqrt{2}$ at $t = 5\pi/4$.

Question: Can you write $f(t)$ in the form

$$f(t) = A\sin(t - \phi)$$

for some constants A, ϕ?

$$f(t) = \sqrt{2}\sin(t + \pi/4)$$

Adding sines and cosines

Challenge: Consider the function

$$f(t) = a \cdot \sin(t) + b \cdot \cos(t)$$

where a and b are two constants. Write $f(t)$ in the form

$$f(t) = A \sin(t - \phi)$$

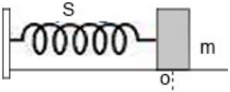
for some constants A, ϕ.

There are two ways to approach this: the sine angle addition formula, OR using inverse trigonometric functions.

Answer: $f(t) = \pm(\sqrt{a^2 + b^2}) \sin(t + \arctan(b/a))$

Oscillating Spring

One end of a spring is attached to a wall, and the other to a moveable mass. The position of the mass is given by $x(t)$ which satisfies the differential equation



$$x'' = kx$$

Question: Is $k > 0$ or $k < 0$? (Think about how springs behave.)

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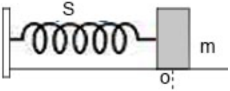
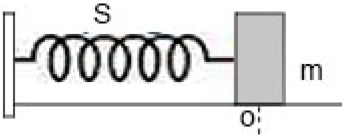


$$x'' = kx$$

Question: Is $k > 0$ or $k < 0$? (Think about how springs behave.) k is negative.

Oscillating Spring

$$x'' = -9x$$



Question: Can you find solutions to this differential equation of the form $x(t) = \sin(\omega t)$ for some frequency ω? What about $x(t) = \cos(\omega t)$?

Oscillating Spring

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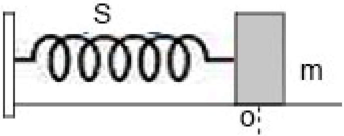

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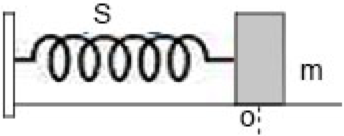
$$x(t) = \sin(3t)$$

$$x(t) = \cos(3t) = \sin(3(t + \frac{\pi}{6}))$$

Oscillating Spring

$$x'' = -9x$$



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$$x(t) = \sin(3t)$$

$$x(t) = \cos(3t) = \sin(3(t + \frac{\pi}{6}))$$

The general solution is

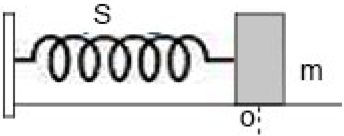
$$x(t) = A\sin(3(t - \phi))$$

where A, ϕ are constants.

Oscillating Spring

$$x'' + 9x = 0$$

$$x(t) = A\sin(3(t - \phi))$$

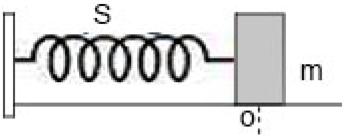


Question: Suppose that the mass oscillates between $x = -2$ and $x = 2$, and suppose that $x(0) = -2$. Calculate $x(t)$.

Oscillating Spring

$$x'' + 9x = 0$$

$$x(t) = A\sin(3(t-\phi))$$



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$x(t)$ has amplitude 2, and has a minimum at $t = -2$. Thus,

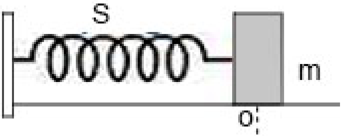
$$x(t) = -2\cos(3t)$$

This can also be written as $x(t) = 2\sin(3(t - \frac{\pi}{6}))$

Oscillating Spring

$$x'' + 9x = 0$$

$$x(t) = x(t) = A\sin(3(t-\phi))$$



Question: Suppose that the mass oscillates between $x = -2$ and $x = 2$, and $x(0) = 1$. Based on this information, there are two possibilities for what $x(t)$ is.

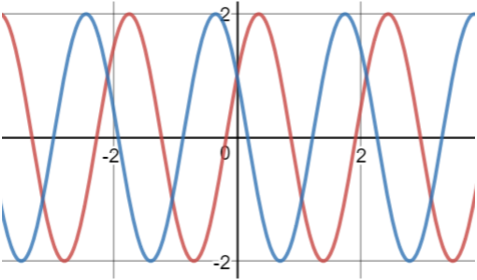
- Suppose that $x'(0) > 0$. Based on this information, sketch a graph of what $x(t)$ looks like. Label the zeroes and CP's. Then use this to express $x(t)$ in the form

$$x(t) = A\sin(3(t-\phi))$$

- Do the same for the case $x'(0) < 0$.

Solution

Here is what the two possible graphs look like, for $x'(0) > 0$ and $x'(0) < 0$.



Solution

We have that the function oscillates between $x = -2$ and $x = 2$. So we can write the function in the form

$$x(t) = 2\sin(3(t - \phi)) = 2\sin(3t - 3\phi)$$

for some phase ϕ. Notice that since $\sin(3t)$ has period $\frac{2\pi}{3}$, we can assume $0 \leq 3\phi < 2\pi$.
Since $x(0) = 1$, we find that $2\sin(-3\phi) = 1$.
Subject to the constraint that $0 \leq 3\phi < 2\pi$, we find that $3\phi = \frac{7\pi}{6}, \frac{11\pi}{6}$. One can check that the first possibility gives the case $x'(0) < 0$, and the second gives the case $x'(0) > 0$.